Mathematical aspects of classical and quantum mechanics.

This is an outline of an online course which will meet once a week on Tuesdays from 19:20 till 20:55, during the Spring semester at the Yau Mathematical Science Center at Tsinghua University. The first meeting is on February 22 (2.22.2022), the zoom link will be announced on the website https://tqfts.com. The meeting ID is 965 0580 6932.

The course has two parts. The first one is a mathematical introduction to classical mechanics. The second one is a mathematical introduction to quantum mechanics. Below is an outline of the material that will be covered in this course.

Part one: classical mechanics.

(1) Classical Newtonian mechanics of particles in a potential force in $\mathbb{R}^3$. Examples: Kepler system, harmonic oscillator. Lagrangian formulation, action functional.
(2) Constrained systems, potentials localized on a submanifold (configuration space) in $\mathbb{R}^3 n$. Lagrangian mechanics on a manifold. Lagrangian of a system as a function on the tangent bundle to the configuration space. Euler-Lagrange equations, boundary conditions. Integrals of motion.
(3) Hamiltonian reformulation. Legendre transform. Phase space as the cotangent bundle to the configuration space. Hamiltonian of a system as a function on the phase space. Hamilton equations. Hamiltonian as conserved quantity. Conservation of energy.
(4) Symplectic structure on the cotangent bundle to a manifold. Digression into symplectic geometry: symplectic manifold, isotropic submanifold, coisotropic submanifold, Lagrangian submanifold. Hamiltonian vector field, trajectories of a Hamiltonian system as flow lines of the Hamiltonian vector field. Examples of symplectic manifolds. Symplectic reduction.
(5) Hamilton-Jacobi action for exact symplectic manifolds. Hamiltonian mechanics on general symplectic manifolds. Example: spheres. Poisson brackets
(6) Summary: Poisson algebra of classical observables. States on the classical algebra of observables. Evolution of observables and evolution of states. Digression: stability, theoretical determinism and practical chaos.

Part two: quantum mechanics.

(1) Mathematical principles of quantum mechanics:
    - Algebra of quantum observables: a family of associative algebras over $\mathbb{C}$ which is a deformation family of the complexified Poisson algebra of classical observables.
    - $*$-structure on the algebra of observables. Observables and states. Representations of the algebra of observables. Pure and mixed states. Density matrix.
    - Matrix quantum mechanics. Quantum uncertainty principle.
    - Examples: quantization of $T^*\mathcal{N}$ and differential operators on $\mathcal{N}$, quantization of Hamiltonian systems on S^2 with its symplectic structure and representation theory of $SU(2)$.
(2) Quantum dynamics. Quantum Hamiltonian, evolution operator as an integral operator, evolution of observables, evolution of states. Scattering amplitudes.

(3) Spectra of observables. Energy spectrum of quantum Hamiltonians. Hydrogen atom, one dimensional Schroedinger operator with rapidly decaying potential.
(4) Evolution operator $e^{i\frac{widehatHt}{h}}$ as an integral operaor. Its semiclassical limit ($h \to 0$). The semiclassical path integral representation of the evolution operator.

Prerequisites. Linear algebra, analysis. Basic familiarity with manifolds will be helpful but is not necessary. I will introduce and recall all basic notions.

Grades. This is a seminar type course. I will assign homework, but it will not be graded. I will post solutions after a couple of weeks.

Website. The website for this course can be found on my research page: https://tqfts.com, where you can find lecture notes materials, links to recorded lectures.

Literature. There are many textbooks on quantum mechanics. They are a good supplementary material. I recommend the book [1] which is more elementary introduction and the book [2] which is more advanced.

References

[1] L.D.Faddeev. O.A. Yakubovskii, "Lectures on Quantum Mechanics for Mathematics Students", AMS, Student Mathematical Library, vol. 47, 2009
[2] L.A. Takhtajan, "Quantum Mechanics for Mathematicians", AMS, Graduate Studies in Mathematics, vol. 95, 2008